A SINGULAR VALUE OF Π.

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On page 291 of Ray's Calculus may be seen a demonstration of the following well known theorem of Wallis:

$$\frac{\pi}{2}=\frac{2.2.4.4.6.6.8.8\ldots\ldots}{1.3.3.5.5.7.7.9\ldots\ldots} \tag{1}$$

By the binomial formula

$$(1-1)^n=1-n+\frac{n(n-1)}{2}-\frac{n(n-1)(n-2)}{2.3}+\frac{n(n-1)(n-2)(n-3)}{2.3.4}\ldots \tag{2}$$

Factoring,

$$(1-1)^n=\frac{(1-n)\ (2-n)\ (3-n)\ (4-n)}{1\ .\ 2\ .\ 3\ .\ 4}\ldots \tag{3}$$

Substituting $-n$ for n,

$$(1-1)^{-n}=\frac{(1+n)\ (2+n)\ (3+n)\ (4+n)}{1\ .\ 2\ .\ 3\ .\ 4}\ldots \tag{4}$$

Multiplying (3) by (4),

$$(1+1)^n(1-1)^{-n}=\frac{(1-n^2)(4-n^2)(9-n^2)(16-n^2)}{1\ .\ 2.2\ .\ 3.3\ .\ 4.4}\ldots \tag{5}$$

Substituting $\frac{1}{2}$ for n, and reducing,

$$(1-1)^{\frac{1}{2}}(1-1)^{-\frac{1}{2}}=\frac{1.3.3.5.5.7.7.9\ldots}{2.2.4.4.6.6.8.8\ldots} \tag{6}$$

Combining (1) and (6),

$$\pi=\frac{2}{(1-1)^{\frac{1}{2}}(1-1)^{-\frac{1}{2}}}.$$

ANSWER TO PROF. SCHEFFER'S QUER (P. 31, VOL. VIII.)

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Query.—"If of any curve we find the evolute, and of the latter the evolute, and so on ad infin., the ultimate evolute is a cycloid. How is this proved?"

Answer.—The proposition stated is not correct.

Let s_0 be the length of the given curve, measured from the origin to any point (the origin being a point on the curve). Let β_0 be the inclination of the tangent at that point to the axis of abscissas, and let R_0 be the radius of curvature at that point. Let s_n, β_n, R_n be like quantities for the corres-

ponding points on the nth evolute of the curve, and s_{-n}, β_{-n}, R_{-n} like quantities for the corresponding point on its nth involute. In this notation I regard the centre of curvature at any point as corresponding to that point.

Then

$$R_0 = \frac{ds_0}{d\beta_0};\ s_1 = C_0 + R_0 = C_0 + \frac{ds_0}{d\beta_0}.$$

But since the directions of a curve and its evolute at the corresponding points are perpendicular, we have

$$\beta_n = \beta_{n-1} - \tfrac{1}{2}\pi,$$

and by differentiating,

$$d\beta_n = d\beta_{n-1} = d\beta_0.$$

By the principles of the foregoing argument

$$s_2 = C_1 + \frac{ds_1}{d\beta_0} = C_1 + \frac{d^2 s_0}{d\beta_0^2},$$

and generally

$$s_n = C^{n-1} + \frac{d^n s_0}{d\beta_0^n},$$

C with its various subscripts signifying constants.

It is then plain that if s_0 be given as a function of β_0, s_n can be at once deduced as a function of β_0, and of course as a function of β_n; and this function depends on the form of the given function, as the arbitrary constant introduced does not affect the form of the evolute. Thus if $s_0 = \beta_0^i$ (i signifying an integer), successive differentiations will at last bring the eq. $s_n = C_{n-1}\,\beta_n$, which is the equation of a circle.

A cycloid does not in general result from taking successive evolutes. If however successive *involutes* be taken, the arbitrary constant introduced by one integration becomes a coefficient of β_0 in the next integration; so that the form of the ultimate involute depends on the arbitrary constants.

If in determining the arbitrary constants we make β_0 successively zero and $\frac{1}{2}\pi$ the successive integrations will bring at last an equation indefinitely approximating

$$C + s_{-4n} = C_{-4n}\cos\beta_0 = -C_{-4n}\cos\beta_{-4n}.$$

This is the equation of a cycloid, and no doubt the proposition intended was this:—

If of any curve we find the *involute*, taking the extreme radii of curvature perpendicular to each other, and of the latter the involute in like manner, and so on ad infinitum, the ultimate involute is a cycloid.